Nonobtuse local tetrahedral refinements towards a polygonal face/interface

Sergey Korotov[1], Michal Křížek[2]

[1] BCAM - Basque Center for Applied Mathematics
Bizkaia Technology Park, Building 500, E–48160 Derio
Basque Country, Spain
e-mail: korotov@bcamath.org
[2] Institute of Mathematics, Academy of Sciences
Žitná 25, CZ–115 67 Prague 1, Czech Republic
e-mail: krizek@math.cas.cz

Abstract: In this note we show how to generate and conformly refine nonobtuse tetrahedral meshes locally in the neigbourhood of a polygonal face or a polygonal interior interface of a three-dimensional domain. The technique proposed can be used for example for problems with boundary and/or interior layers, and for interface problems.

Keywords: finite element method, nonobtuse tetrahedron, local refinement, discrete maximum principle, boundary and interior layers, interface problem

Mathematical Subject Classification: 65M50, 65N30, 65N50

1 Introduction

Nonobtuse simplicial elements play an important role in the finite element analysis of boundary value problems, since they yield irreducible and diagonally dominant stiffness matrices for sufficiently small discretization parameter and guarantee the validity of the discrete maximum principle when solving many boundary value elliptic problems (see [2, 3]). Note that one obtuse simplex in a triangulation can destroy the discrete maximum principle [2]. In [4], we gave a global refinement algorithm which produces nonobtuse tetrahedra. However, various local refinements of simplicial meshes are often necessary to handle e.g. boundary or interior layers, various oscillations and singularities of the solution or its derivatives at interior interfaces, where one kind of media changes into another, or at some special points, see [7]. One such an algorithm for treating vertex (or point) singularities was first presented in [5], and then generalized in [1]. Edge and face singularities can also be treated by that algorithm if we select sufficiently many additional vertices along edges or near faces. In this note we present another algorithm for a face-to-face tetrahedral refinement towards a flat face (or interface) of a three-dimensional domain which yields only nonobtuse tetrahedra.

Recall that a tetrahedron is said to be *nonobtuse*, if all its six dihedral angles between faces are nonobtuse.

In Figure 1 we observe several examples of basic nonobtuse tetrahedra, namely, the path-tetrahedron, the cube corner tetrahedron, and the regular tetrahedron (see [2] for their definitions). Notice that edges forming right angles in triangular faces in the cases a) and b) are not necessarily of the same length.

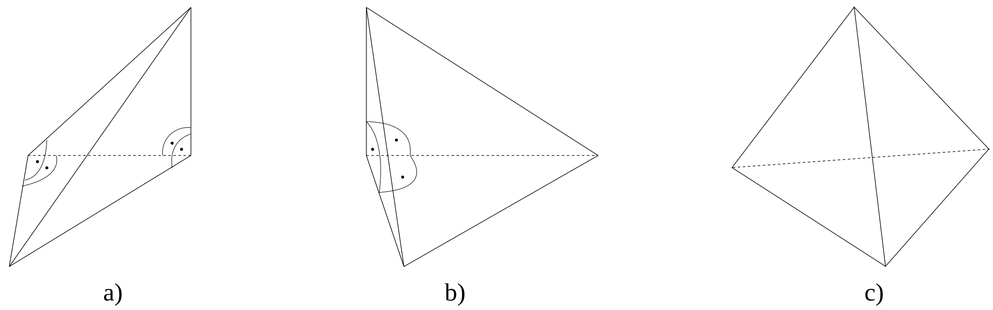

Figure 1: Examples of nonobtuse tetrahedra: a) path, b) cube corner, and c) regular.

2 The main idea

In this section we give a key idea and also an illustration (see Figures 2 and 3) of the nonobtuse tetrahedral refinements towards a flat face of (or interface inside) the 3D solution domain. For this purpose we take a square prism (e.g. a cube) and its adjacent square prism. Denote their vertices and some other nodes as sketched in Figure 2, where also partitions of some faces are given.

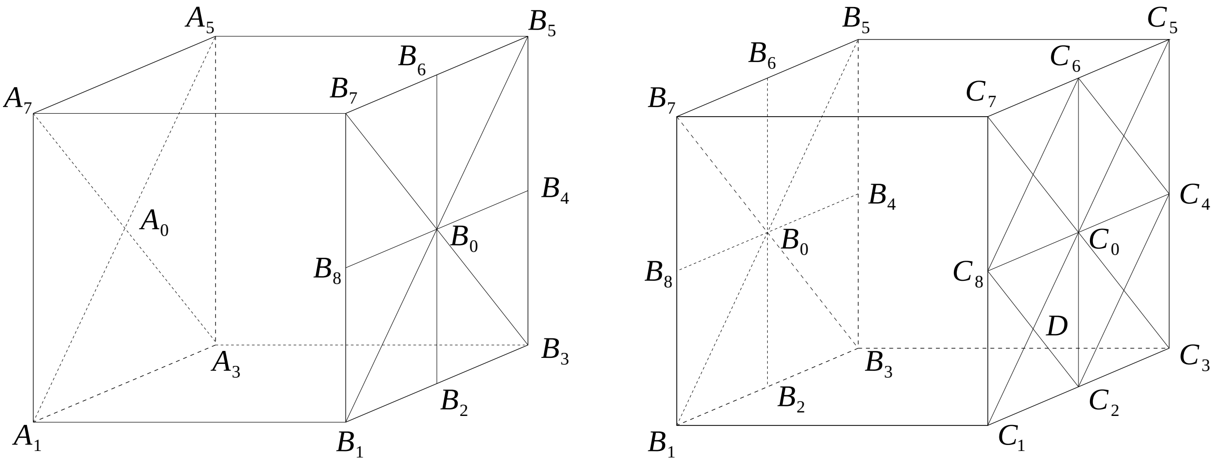

Figure 2: A sketch of a decomposition of two adjacent square prisms into nonobtuse tetrahedra.

In what follows, let $d = |B_1B_3| = |B_3B_5|$ denote the length of sides of the square faces of the considered prisms, and let $l_1 = |A_0B_0|$ and $l_2 = |B_0C_0|$ be their thicknesses.

First we decompose the left square prism $A_1A_3A_5A_7B_1B_3B_5B_7$ of Figure 2 into four triangular prisms whose common edge is A_0B_0. Second we decompose each triangular prism into four tetrahedra. For instance, the triangular prism $A_0A_1A_3B_0B_1B_3$ will be divided in the following way (see Figure 3):

$A_0A_1A_3B_0$ (cube corner tetrahedron), $A_1B_1B_2B_0$ (path tetrahedron),
$A_3B_3B_2B_0$ (path tetrahedron), and $A_1A_3B_0B_2$.

The first three resulting tetrahedra are clearly nonobtuse. The last tetrahedron $A_1A_3B_0B_2$ is the union of two path tetrahedra whose common face is $A_2B_0B_2$, where A_2 is the midpoint of A_1A_3. We see that $A_1A_3B_0B_2$ is nonobtuse if and only if

$$|B_1B_3| \leq 2|A_0B_0|, \quad \text{i.e.} \quad l_1 \geq \frac{d}{2}. \tag{1}$$

The other three triangular prisms, $A_0A_3A_5B_0B_3B_5$, $A_0A_5A_7B_0B_5B_7$, and $A_0A_1A_7B_0B_1B_7$, can be subdivided similarly.

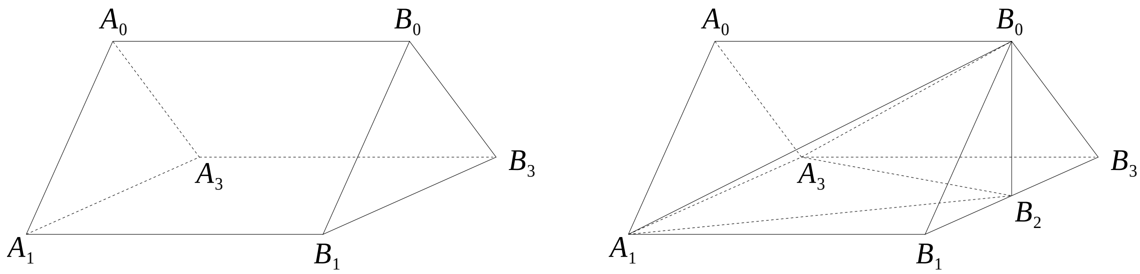

Figure 3: Decomposition of a triangular prism $A_0A_1A_3B_0B_1B_3$ into four tetrahedra.

Next, we decompose the right adjacent square prisms $B_1B_3B_5B_7C_1C_3C_5C_7$ of Figure 2 into eight triangular prisms whose common edge is B_0C_0. Further, e.g., the triangular prism $B_0B_1B_2C_0C_1C_2$ will be divided into four tetrahedra like in the previous step:

$B_0B_1B_2C_2$ (cube corner tetrahedron), $B_0C_0DC_2$ (path tetrahedron),
$B_1C_1DC_2$ (path tetrahedron), and $B_0B_1DC_2$.

The last tetrahedron is nonobtuse provided

$$|B_0B_1| \leq 2|B_0C_0|, \quad \text{i.e.} \quad l_2 \geq \frac{\sqrt{2}d}{4}. \tag{2}$$

This condition is necessary and sufficient to guarantee a nonobtuse decomposition of the triangular prism $B_0B_1B_2C_0C_1C_2$ into four nonobtuse tetrahedra as described above.

The other seven triangular prisms can be divided into nonobtuse tetrahedra similarly. In this way (i.e., under conditions (1) and (2)) we obviously get a face-to-face nonobtuse partition of two adjacent square prisms. The left square prism of Figure 2 is subdivided into 16 and the right prism into 32 nonobtuse tetrahedra. This enables us to form layers and use this process repeatedly.

3 Examples and remarks

In this section we shall give several examples of how to apply the ideas of Section 2 in some concrete situations.

Example 1 We shall demonstrate how to use the above-described algorithm recursively for the square prism $\mathcal{P} = [0, p+q] \times [0, 1] \times [0, 1]$ (see Figure 4), where

$$p \geq 1 \quad \text{and} \quad q \geq \frac{\sqrt{2}}{2} \tag{3}$$

are real parameters.

First we decompose $\mathcal{P}$ into infinitely many square prisms (layers) $P_n = [x_n, x_{n+1}] \times [0, 1] \times [0, 1]$, $n = 1, 2, 3\ldots$, where

$$x_1 = 0, \; x_2 = x_1 + \frac{p}{2}, \; x_3 = x_2 + \frac{q}{2}, \; x_4 = x_3 + \frac{p}{4}, \; x_5 = x_4 + \frac{q}{4}, \tag{4}$$

etc. It is clear that $x_n \to p + q$ as $n \to \infty$.

Now, consider the first two prisms P_1 and P_2, which, in terms of Figure 2, have $d = 1$, and thicknesses $l_1 = \frac{p}{2} = \frac{pd}{2}$ and $l_2 = \frac{q}{2} = \frac{qd}{2}$, respectively. Under the prescribed conditions (3) for p and q, relations (1) and (2) are obviously valid, and therefore, prisms P_1 and P_2 can be decomposed into nonobtuse tetrahedra as described in Section 2.

The next prisms P_3 and P_4 will be first subdivided into four smaller square prisms each, i.e. now $d = \frac{1}{2}$. After that we again use the algorithm as marked in Figure 2, but now for four pairs of double smaller square prisms, for which "their" d, l_1, and l_2 are just twice smaller than those for the previous two prisms P_1 and P_2. As conditions (1) and (2) are valid again for all pairs of smaller prisms, we get a partition of prisms P_3 and P_4 into nonobtuse tetrahedra. Clearly, the total conformity of such "advancing" tetrahedral mesh is provided and we can generate infinitely many refinement steps in this manner.

However, in real-life calculations, we are only interested in the finite partitions. Therefore, after refining n layers as described above, we should finish with a suitable nonobtuse partition of the remaining square prism $[x_{n+1}, p+q] \times [0, 1] \times [0, 1]$. It can be decomposed into nonobtuse tetrahedra in many ways, e.g. into path-tetrahedra (cf. Figure 4).

Remark 1 Notice that for all generated layers the inequalities in (1) and (2) become equatilies if $p = 1$ and $q = \sqrt{2}/2$ in the above example. In this special case this refinement process yields only the so-called ortho-tetrahedra which have three orthogonal edges [2]. For $p > 1$ or $q > \sqrt{2}/2$ some tetrahedra will be not ortho-tetrahedra, but they will remain nonobtuse.

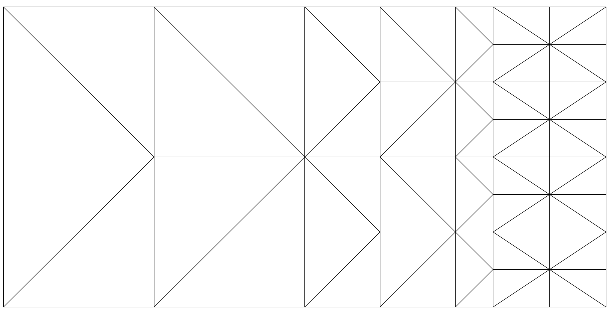

Figure 4: A partition of a face of $\mathcal{P}$ which is parallel with the x-axis in Example 1 for $p = q = 1$ and $n = 6$.

Example 2 Consider the unit cube $\mathcal{C} = [0,1] \times [0,1] \times [0,1]$. We can generate nonobtuse simplicial refinements towards its right face as follows: take $q = \sqrt{2}/2$ (the quotient of a geometric sequence). This choice is quite natural to get nondegenerated tetrahedra due to relations (1) and (2). Let $x_1 = 0$, $x_2 = \frac{4-5q}{4-4q}$, and for $n = 3, 4, \ldots$ we set

$$x_n = x_2 + \frac{q}{4} \sum_{i=3}^{n} q^{i-3}. \tag{5}$$

Then we see that

$$x_n \to \frac{4-5q}{4-4q} + \frac{q}{4}\frac{1}{1-q} = 1 \quad \text{as } n \to \infty.$$

This enables us to define square prisms $Q_n = [x_n, x_{n+1}] \times [0,1] \times [0,1]$ for $n = 1, 2, \ldots$. Each of the first two prisms Q_1 and Q_2 will be first subdivided into four square subprisms and then decomposed into nonobtuse tetrahedra like in previous Example 1. The thickness of Q_1 is x_2 and of Q_n is $q^{n-1}/4$ when $n > 1$. For $k = 1, 2, \ldots$ we set $d = 2^{-k}$. If $n = 2k-1$ then n is odd and we have

$$l_1 = \frac{q^{n-1}}{4} = \frac{1}{2^{k+1}} = \frac{d}{2} \quad \text{for } k > 1$$

and $l_1 = x_2 > \frac{d}{2} = \frac{1}{4}$, i.e., (1) is valid. If $n = 2k$ then n is even and we obtain

$$l_2 = \frac{q^{n-1}}{4} = \frac{\sqrt{2}}{2}\frac{1}{2^{k+1}} = \frac{\sqrt{2}d}{4} \quad \text{for } k = 1, 2, \ldots,$$

i.e., (2) is valid as well and all triangular prisms from $Q_2, Q_3, \ldots$ have the same shape up to scaling.

We again perform only a finite number of refinement steps and generate prisms $Q_1, Q_2, \ldots, Q_n$. According to (5), the thickness of the remaining square prism $[x_{n+1}, 1] \times [0,1] \times [0,1]$ is $\frac{q^{n-1}}{4}\frac{1}{1-q}$ which is three times greater than $q^{n-1}/4$. Hence, by (1) and (2) the remaining prism can be decomposed face-to-face into nonobtuse tetrahedra.

Example 3 We can generate nonobtuse tetrahedral refinements for even more general three-dimensional domains. For instance, Figure 5 shows a polygonal face (marked by bold line) that also allows nonobtuse refinements in the above presented manner in its neighbourhood.

Remark 2 Notice that nonobtuse tetrahedral meshes (whose elements have nonobtuse triangular faces [2]) satisfy the maximum angle condition [6], which is one of sufficient conditions for convergence proofs in the finite element analysis.

Acknowledgement: This paper was supported by Grant MTM2008-03541 of the MICINN, Spain, the ERC Advanced Grant FP7-246775 NUMERIWAVES, Grant PI2010-04 of the Basque Government, Institutional Research Plan nr. AV0Z 10190503 of the Academy of Sciences of the Czech Republic, and Grant nr. IAA 100190803 of the Academy of Sciences of the Czech Republic. The authors are also indebted to Miloslav Feistauer for inspiration.

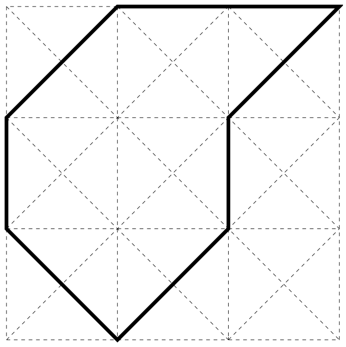

Figure 5: Illustration for Example 3.

References

[1] Brandts, J., Korotov, S., Křížek, M., Dissection of the path-simplex in R^n into n path-subsimplices, *Linear Algebra Appl.* 421 (2007), 382–393.

[2] Brandts, J., Korotov, S., Křížek, M., Šolc, J., On nonobtuse simplicial partitions, *SIAM Rev.* 51(2) (2009), 317–335.

[3] Karátson, J., Korotov, S., Discrete maximum principles for FEM solutions of some nonlinear elliptic interface problems, *Internat. J. Numer. Anal. Model.* 6 (2009), 1–16.

[4] Korotov, S., Křížek, M., Acute type refinements of tetrahedral partitions of polyhedral domains, *SIAM J. Numer. Anal.* 39 (2001), 724–733.

[5] Korotov, S., Křížek, M., Local nonobtuse tetrahedral refinements of a cube, *Appl. Math. Letters* 16 (2003), 1101–1104.

[6] Křížek, M., On the maximum angle condition for linear tetrahedral elements, *SIAM J. Numer. Anal.* 29 (1992), 513–520.

[7] Roos, H.-G., Stynes, M., Tobiska, L., Robust numerical methods for singularly perturbed differential equations. Springer Series in Comput. Math. vol. 24, Springer-Verlag, Berlin, Heidelberg, 2008.